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Algorithms & Data Structures — Homework 4 — HS 18

Exercise Class (Room & TA): ______________________

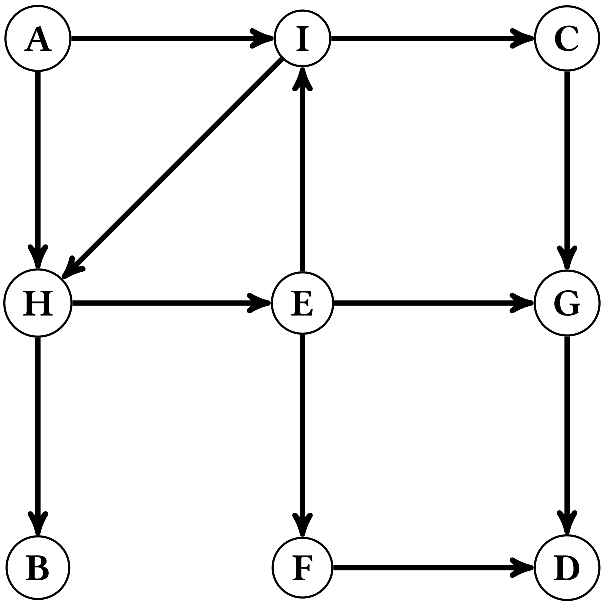
Submitted by: ______________________

Peer Feedback by: ______________________

Points: _____

Exercise 4.1 *Depth-First Search.*

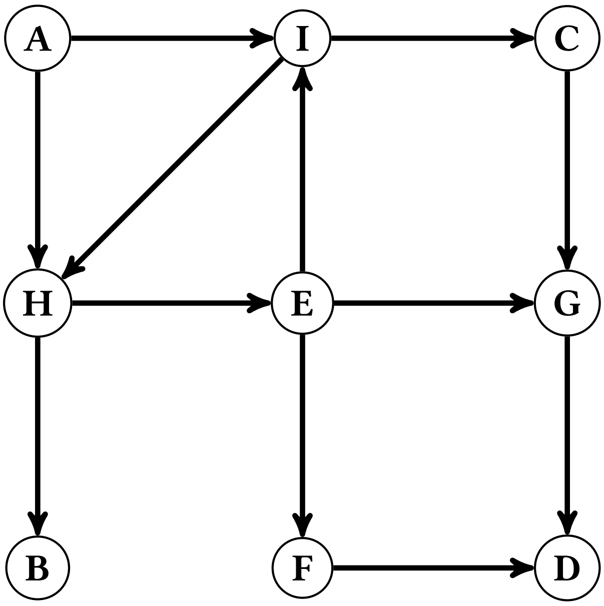
Execute a depth-first search (Tiefensuche) on the following graph starting from vertex A (using a stack, as seen in lecture). Assume that we push successor vertices (Nachfolger) on the stack in *reverse* alphabetical order. (For example, if the successors are R and U, then we first U push on the stack and then R.)



1. Mark the edges that belong to the depth-first tree (Tiefensuchbaum) with a “T” (for tree edge).
2. For each vertex, give its *pre*- and *post*-number.
3. Mark every forward edge (Vorwärtskante) not in the depth-first tree with an “F”, every backward edge (Rückwärtskante) with an “B”, and every cross edge (Querkante) with a “C”.
4. Has the above graph a topological ordering? How can we use the above execution of depth-first search in order to see this?

Exercise 4.2 *Breadth-First Search* ***(2 Points).***

On the following graph, execute a breadth-first search (Breitensuche) starting from vertex A (using a queue, as seen in the lecture). Assume that successor vertices (Nachfolger) are enqueued in alphabetical order.



1. Write down the order in which the vertices are dequeued during this execution of breadth-first search.

2. As seen in the lecture, breadth-first search can be used to determine the distances for all vertices from the start vertex. These distances partition the graph into level sets $L_0, L_1, L_2, \ldots$, where L_i is the set of all vertices with distance i from the start vertex.

   Use the above execution of breadth-first search to compute the distances from the start vertex and write down these level sets.

3. Consider the following questions about level sets $L_0, L_1, \ldots$ computed by breadth-first search in directed and undirected graphs. Justify your answer.

   - In a directed graph, can there be an edge from a level set L_i with $i \geq 2$ to a level set L_j with $j \leq i - 2$?
   - In an undirected graph, can there be an edge from a level set L_i with $i \geq 2$ to a level set L_j with $j \leq i - 2$?
   - In a directed graph, can there be an edge from a level set L_i with $i \geq 0$ to a level set L_j if $j \geq i + 2$?

4. Let G be a connected undirected graph, let s be a vertex in G, and let $L_0, L_1, \ldots$ be the level sets computed by breadth-first search starting from vertex s. Prove that G is bipartite if and only if there are no edges between two vertices in the same level set L_i.

Exercise 4.3 *Asymptotic Notation.*

1. Suppose f satisfies the condition $f(n) \geq 1$ for all $n \geq 1$. Show that if $g \leq O(f)$, then for every $D \geq 0$, we have $g(n) + D \leq O(f(n))$.
2. Let $f(n) = \frac{1}{n}$, and $g(n) = f(n) + 1$. Does $g(n) \leq O(f(n))$ hold? Justify your answer.
3. In class, we defined $O(f)$ to consist of all functions $g(n)$ such that

$$\exists C > 0.\ \forall n \geq 1.\ g(n) \leq C \cdot f(n)\,.$$

   Another definition for $O(f)$, commonly found in the literature, includes all functions that satisfy the a-priori weaker condition,

$$\exists C > 0.\ \exists n_0 \geq 1.\ \forall n \geq n_0.\ g(n) \leq C \cdot f(n)\,.$$

   (This condition is a-priori weaker, because it requires the inequality $g(n) \leq C \cdot f(n)$ to hold only for all $n \geq n_0$ instead of for all $n \geq 1$.)

   Prove that the two definitions of $O(f)$ are in fact equivalent if the function f satisfies $f(n) > 0$ for all $n \geq 1$ (which is typically the case for functions that arise as running times of algorithms).
4. Show that, if we don't require f to satisfy the condition $f(n) > 0$ for all $n \geq 1$, the above two definitions of $O(f)$ are not necessarily equivalent.

   Provide concrete functions f and g such that g satisfies the second definition of $O(f)$ but not the first.

Exercise 4.4 *Pouring water **(1 Point)**.*

We have three containers whose sizes are 15 liters, 9 liters, and 5 liters, respectively. The 15-liter container starts out full of water, but the 9-liter and 5-liter containers are initially empty. We are allowed one type of operation: pouring the contents of one container into another, stopping only when the source container is empty or the destination container is full. We want to find a shortest sequence of pourings that leaves exactly 2 liters in one of the containers.

1. Model this as a graph problem: give a precise definition of the graph involved and state the specific question about this graph that needs to be answered.
2. Find a shortest sequence of pourings which leaves exactly 2 liters in one of the containers. Prove that this sequence is actually shortest.